

The σ -Model and Non-commutative Geometry ¹

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Abstract

In terms of non-commutative geometry, we show that the σ -model can be built up by the gauge theory on discrete group Z_2 . We introduce a constraint in the gauge theory, which lead to the constraint imposed on linear σ model to get nonlinear σ model .

1 Introduction

It has become widely accepted that strong interactions exhibit a set of approximate symmetries corresponding the chiral groups $SU_L(2) \times SU_R(2)$ and $SU_L(3) \times SU_R(3)$. The best known Lagrangian model based on $SU_L(2) \times SU_R(2)$ is the so called σ -model[1]. There is a quadruplet of real fields in the linear σ model. However, the particle corresponding one of them has not been observed experimentally. To meet this difficulty, a constraint has to be imposed to eliminate this particle, then the nonlinear σ model was introduced. So far, nonlinear σ models have widespread applications in statistical mechanics as well as in quantum field theory. Whether there is any profound meaning in σ -model from the ordinary geometrical point of view is an open question.

¹Work supported in part by The National Natural Science Foundation Of China.

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Since Alain Connes[2] applied his non-commutative geometry to the particle physics model building in which the Higgs fields were introduced as gauge fields, many efforts have been done along similar direction[3-6]. In particular, Sitarz[7] developed discrete points idea and built a gauge theory on discrete group. Soon after, the physical model building along this direction was completed by the authors[8]. However, the relationship between sigma model and non-commutative geometry has never been explored.

In this letter, we show that the sigma model may be regarded as a kind of gauge field in non-commutative geometry, based on the Yang-Mills like gauge theory on discrete group developed by the present authors[8]. The constraint imposed on the linear σ model is introduced naturally by the gauge theory, which may explore why nonlinear σ model is the realistic model.

2 Differential Calculus on Discrete Group

In this section, we will outline the notion of differential calculus theory on discrete groups. For details, it is referred to [7].

Let G be a discrete group of size N_G and \mathcal{A} the algebra of complex valued functions on G . The right and left multiplications of G induce natural automorphisms of \mathcal{A} . The right and left actions on \mathcal{A} read

$$(R_h f)(g) = f(g \odot h), \quad (L_h f)(g) = f(h \odot g), \quad (1)$$

where \odot denotes the group multiplication. The derivative on \mathcal{A} is defined as the left (or right) invariant vector space which satisfies the following condition

$$\partial \in \mathcal{F} \Leftrightarrow \{L_h \partial(f) = \partial(L_h f), \quad \forall f \in \mathcal{A}\}. \quad (2)$$

The basis of \mathcal{F} , ∂_i , $i = 1, \dots, N_G$ is given by

$$\partial_g f = f - R_g f, \quad g \in G', f \in \mathcal{A}, \quad (3)$$

where $G' = G \setminus e$, which is composed by the elements of group G eliminated the unit e . As, ∂_e , e the unit of G , is trivial, $\partial_e f = 0$, $\forall f \in \mathcal{A}$, the dimension of \mathcal{F} is then $N_G - 1$. \mathcal{F} forms an algebra with relations

$$\partial_i \partial_j = \sum_k C_{ij}^k \partial_k, \quad (4)$$

where C_{ij}^k are the structure constants satisfying

$$\sum_l C_{ij}^l C_{lk}^m = \sum_l C_{il}^m C_{jk}^l$$

due to the associativity of the algebra \mathcal{F} . It is easy to check the following identity

$$\partial_g \partial_h = \partial_g + \partial_h - \partial_{h \odot g} (1 - \delta_{h \odot g}^e) \quad g, h \in G',$$

with structure constants $C_{gh}^g = 1$, $C_{gh}^h = 1$, $C_{gh}^{h \odot g} = -(1 - \delta_{h \odot g}^e)$. In the case of $Z_2 = \{e, r\}$, for example, the only nontrivial structure constant is $C_{rr}^r = 2$.

The Haar integral is introduced as a complex valued linear functional on \mathcal{A} that remains invariant under the action of R_g ,

$$\int_G f = \frac{1}{N_G} \sum_{g \in G} f(g), \quad (5)$$

which is normalized such that $\int_G 1 = 1$.

3 Yang-Mills Like Gauge Theory on Discrete Group

In this section, we introduce a free fermion Lagrangian on $M \times Z_2$ and build a Yang-Mills like gauge theory on $M \times Z_2$, in which a simple complex Higgs field is the gauge field with respect to Z_2 -gauge symmetry and the Yukawa couplings between the fermions and Higgs is obtained by the covariant derivatives or the minimum gauge coupling.

To build the Yang-Mills like gauge theory, let us first assign the free fermion field with respect to Z_2 elements as follow

$$\psi(x, e) = \sqrt{2} \psi_L(x), \quad \psi(x, r) = \sqrt{2} \psi_R(x), \quad e, r \in Z_2, \quad (6)$$

where

$$\psi_L(x) = \frac{1}{2}(1 + \gamma_5)\psi(x), \quad \psi_R(x) = \frac{1}{2}(1 - \gamma_5)\psi(x)$$

are the left and right handed field respectively.

If we introduce a Lagrangian on discrete group Z_2 as follow

$$\mathcal{L}(x, h) = \bar{\psi}(x, h)(i\gamma^\mu \partial_\mu + \mu \partial_r)\psi(x, h), \quad h \in Z_2, \quad (7)$$

we find that:

$$\mathcal{L}(x) = \int_{Z_2} \mathcal{L}(x, h) = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - \mu)\psi(x)$$

is just the Lagrangian of free fermion in space time M^4 , then we call the Lagrangian in (7) the free fermion Lagrangian on $M^4 \times Z_2$.

Similar to the reason that leads to the introduction of Yang-Mills fields, it is reasonable to require that the Lagrangian (7) be invariant under gauge transformations $U(h)$, $h \in Z_2$,

$$\psi(x, h) \rightarrow \psi(x, h)' = U(h)\psi(x, h), \quad (8)$$

namely, we require Z_2 symmetry be gauged. Generally speaking, U should be functions on M^4 and discrete group Z_2 , to deal with the sigma model we confine our attention to the case that they are functions on discrete group.

Then for the second term $\mu \overline{\psi(x, h)} \partial_r \psi(x, h)$ in (7), it is needed to introduce gauge covariant derivative D_r to replace ∂_r , which satisfy

$$D_r \psi(x, h) \rightarrow [D_r \psi(x, h)]' = U(h) D_r \psi(x, h), \quad (9)$$

in order that $\bar{\psi}(x, h) D_r \psi(x, h)$ is Z_2 -gauge invariant. This can be realized if we introduce a field $\phi(x, h)$, the Higgs field, as a connection with respect to the Z_2 -gauge symmetry and form the covariant derivative as follows

$$D_r \psi = (\partial_r + \frac{\lambda}{\mu} \phi R_r) \psi. \quad (10)$$

Then the transformation law (9) is satisfied if the generalized gauge field $\phi(x, h)$ has the transformation property

$$\frac{\mu}{\lambda} - \phi' = U(\frac{\mu}{\lambda} - \phi)(R_r U^{-1}). \quad (11)$$

In ordinary Yang-mills gauge theory, we know that we can always choose a gauge at any point to make where the gauge potential vanish. There is a theorem in the differential geometry: For a connection in vector bundle, we can choose a local frame at any point to make the connection vanish at this point.

Accordingly, it is reasonable to require gauge field ϕ may be transformed into zero at any point by choosing a special gauge. In view of these consideration and transformation rule (11), we obtain a constraint on the gauge field as following:

$$\phi = \frac{\mu}{\lambda}(1 - U'^{-1}R_rU'). \quad (12)$$

where U' is an a element of the gauge group. We will find that this constraint is useful in discussing the form of gauge field.

We introduce a new field $\Phi = \frac{\mu}{\lambda} - \phi$ such that the transformation rule (11) becomes

$$\Phi \rightarrow \Phi' = U\Phi(R_rU^{-1}). \quad (13)$$

and the new field Φ has the form as

$$\Phi = \frac{\mu}{\lambda}U'^{-1}R_rU'. \quad (14)$$

Similar to the usual gauge theory where the covariant derivative is equivalent to the covariant exterior derivative $D_M = d_M + igA_\mu dx^\mu$ and $D_M f = D_\mu f dx^\mu$, for the case in hand, the covariant exterior derivative takes form

$$D_{Z_2} = d_{Z_2} + \frac{\lambda}{\mu}\phi\chi^r. \quad (15)$$

The reason is that

$$(d_{Z_2} + \frac{\lambda}{\mu}\phi\chi^r)f = (\partial_r + \frac{\lambda}{\mu}\phi R_r)f\chi^r = D_rf\chi^r.$$

Thus from (7), it follows the generalized gauge invariant Lagrangian for fermions in each sector characterized by Z_2

$$\begin{aligned} \mathcal{L}_F(x, h) &= \overline{\psi(x, h)}\{i\gamma^\mu\partial_\mu + (\mu\partial_r + \lambda\phi(x, h)R_r)\}\psi(x, h) \\ &= \overline{\psi}(x, h)(i\gamma^\mu\partial_\mu + \mu D_r)\psi(x, h). \end{aligned} \quad (16)$$

The Hermitian property of operator ϕR_r requires that

$$\phi^\dagger(x, e) = R_r \phi(x, e) = \phi(x, r). \quad (17)$$

After integrating the last terms in $\mathcal{L}(x, h)$ over Z_2 and in terms of Φ , we get

$$\int_{Z_2} \mu \psi(x, h) D_r \psi(x, h) = -\lambda \bar{\psi}_L(x) \Phi(x) \psi_R(x) - \lambda \bar{\psi}_R(x) \Phi^\dagger(x) \psi_L(x) \quad (18)$$

which is nothing but the Yukawa couplings between the Higgs and chiral fermions.

From direct calculation, similar to Yang-Mills gauge theory, $F_{r\mu}$ and F_{rr} are related to the covariant derivatives respectively

$$\begin{aligned} [D_r, \partial_\mu] \psi &= \frac{\lambda}{\mu} F_{r\mu} R_r \psi = -\frac{\lambda}{\mu} F_{\mu r} R_r \psi, \\ (D_r D_r - 2D_r) \psi &= \frac{\lambda^2}{\mu^2} F_{rr} \psi. \end{aligned} \quad (19)$$

where

$$F_{r\mu} = \partial_\mu \Phi, \quad F_{rr} = \Phi \Phi^\dagger - \frac{\mu^2}{\lambda^2}.$$

Under gauge transformation (8), it is easy to see that

$$F'_{r\mu} = U F_{r\mu} (R_r U^{-1}), \quad F'_{rr} = U F_{rr} U^{-1}.$$

Then we can write down the gauge invariant Lagrangian for the Higgs field as following:

$$\mathcal{L}_H(x, h) = \text{Tr} F_{\mu r} F_r^{\mu\dagger} - \eta \text{Tr} F_{rr} F_{rr}^\dagger. \quad (20)$$

where η is a positive real constant. All the above identities can be derived by non-commutative geometry[7,8]

Therefore, from (16) and (20), we get the entire Lagrangian of the system

$$\mathcal{L}(x) = \int_{Z_2} (\mathcal{L}_F(x, h) + \mathcal{L}_H(x, h)) = \mathcal{L}_F + \mathcal{L}_H \quad (21)$$

where \mathcal{L}_F is gauge invariant part for fermions with Yukawa couplings to Higgs and \mathcal{L}_H is the one for Higgs field

$$\begin{aligned} \mathcal{L}_F &= \overline{\psi(x)} i \gamma^\mu \partial_\mu \psi(x) - \lambda \bar{\psi}_L(x) \Phi(x) \psi_R(x) - \lambda \bar{\psi}_R(x) \Phi(x)^\dagger \psi_L(x), \\ \mathcal{L}_H &= \text{Tr}(\partial_\mu \Phi(x)) (\partial^\mu \Phi(x))^\dagger - \eta \text{Tr}(\Phi(x) \Phi(x)^\dagger - \frac{\mu^2}{\lambda^2})^2. \end{aligned} \quad (22)$$

In this work, the most important idea is that the Yukawa coupling is introduced as gauge coupling. In doing so, Higgs field exists even in the absence of Yang-Mills fields. As an example, we will study σ model in next section.

4 $SU_L(2) \times SU_R(2)$ σ model

For the nucleon doublet field $N = \begin{pmatrix} p \\ n \end{pmatrix}$, we set

$$\psi(x, e) = \sqrt{2}N_L, \quad \psi(x, r) = \sqrt{2}N_R. \quad (23)$$

and require Lagrangian

$$\mathcal{L}(x, h) = \bar{\psi}(x, h)(i\gamma^\mu \partial_\mu + \mu \partial_r)\psi(x, h), \quad h \in Z_2$$

is invariant under gauge transformations

$$\psi(x, h) \rightarrow \psi(x, h)' = U(h)\psi(x, h), \quad (24)$$

where $U(e) \in SU_L(2)$ and $U(r) \in SU_R(2)$ are global to the space time, but are local to the discrete group.

It is clear that the elements of gauge group are independent on the space time but they are functions on the discrete group, therefore, the gauge invariant requires the scalar field Φ must exist. From equation (21), we have the lagrangian for this model as following:

$$\begin{aligned} \mathcal{L} = & i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - \lambda \bar{\psi}_L(x)\Phi(x)\psi_R(x) - \lambda \bar{\psi}_R(x)\Phi^\dagger(x)\psi_L(x) \\ & + \partial_\mu \Phi(\partial^\mu \Phi)^\dagger - \eta(\Phi\Phi^\dagger - \frac{\mu^2}{\lambda^2})^2 \end{aligned} \quad (25)$$

From eq(14), we obtain a constraint on the gauge field as following:

$$\Phi = vSU_L(2)^{-1}SU_R(2) \quad (26)$$

where $v = \frac{\mu}{\lambda}$. We know that elements of $SU(2)$ group may be written as

$$G = \eta + i\tau \cdot \epsilon, \forall G \in SU(2) \quad (27)$$

where $\tau_i, i = 1, 2, 3$ are three Pauli matrices, η, ϵ are real and satisfy the unimodular condition $\eta^2 + |\epsilon|^2 = 1$. Then we may have the form of gauge field Φ as follow:

$$\Phi = vSU_L^{-1}(2)SU_R(2) = \hat{\eta} + i\tau \cdot \hat{\epsilon} \quad (28)$$

where $\hat{\eta}, \hat{\epsilon}$ are real who satisfy condition $\hat{\eta}^2 + |\hat{\epsilon}|^2 = v^2$.

From above arguments, we know that the gauge field Φ valued on a sphere surface of $U(2)$ group and the radius of the sphere is $v = \frac{\mu}{\lambda}$. Hence, we can write Φ as 2×2 matrix

$$\Phi(x) = \sigma I + i\tau^i \pi_i \quad (29)$$

where σ and π are real scalar fields and satisfy a constraint

$$\sigma^2 + |\pi|^2 = v^2. \quad (30)$$

It is obvious that this constraint is remained under the gauge transformation. In following, we will show that if this constraint does not be taken into account, the linear σ model may be obtained, otherwise the nonlinear σ model may be reached.

Rewrite the Lagrangian (25) in terms of the fields σ, π , we get

$$\mathcal{L} = i\bar{N}\gamma^\mu \partial_\mu N - \lambda\bar{N}(\sigma - i\gamma^5 \tau \cdot \pi)N + (\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2 - \eta(\sigma^2 + \pi^2 - \frac{\mu^2}{\lambda^2})^2. \quad (31)$$

The next step is to discuss the transformation property of the fields. If desired one can introduce the infinitesimal gauge transformation as follows,

$$U(h) = 1 + if(h) \cdot \tau, \quad f \in \mathcal{A}, \quad h \in Z_2.$$

and set $f(e) = \alpha, f(r) = \beta$. Explicitly, we have

$$U(e) = SU_L(2) = 1 + i\alpha \cdot \tau, \quad U(r) = SU_R(2) = 1 + i\beta \cdot \tau. \quad (32)$$

Under the gauge transformation (32), from (24) and (13) we have

$$\begin{aligned} N &\rightarrow N + i\frac{\alpha+\beta}{2} \cdot \tau N + i\frac{\alpha-\beta}{2} \cdot \tau \gamma^5 N \\ \sigma + i\pi \cdot \tau &\rightarrow \sigma - (\alpha + \beta) \cdot \pi + i[\pi - \sigma(\alpha + \beta) - (\alpha - \beta) \wedge \pi] \cdot \tau \end{aligned} \quad (33)$$

Redefine the parameter as $\alpha' = \alpha + \beta, \beta' = \beta - \alpha$, we have

$$\begin{aligned} N &\rightarrow N + i\frac{\alpha'}{2} \cdot \tau N - i\frac{\beta'}{2} \cdot \tau \gamma^5 N \\ \sigma &\rightarrow \sigma - \alpha' \cdot \pi \\ \pi &\rightarrow \pi - \sigma \alpha' + \beta' \wedge \pi \end{aligned} \tag{34}$$

which is just those for linear σ model.

In realistic model the constraint (30) has to be required, we can use it to eliminate the σ field from the Lagrangian (31), obtaining the nonlinear σ model,

$$\mathcal{L} = i\bar{N}\gamma^\mu\partial_\mu N - \lambda\bar{N}(\sqrt{v^2 - \pi^2} - i\gamma^5\tau \cdot \pi)N + (\partial_\mu\pi)^2 + \frac{(\pi \cdot \partial\pi)^2}{v^2 - \pi^2} \tag{35}$$

In modern field theory, the problem with the linear sigma model is the particle corresponding to the field σ , which has not been observed experimentally. One possibility to get rid of this particle is to realize the chiral group $SU(2)$ nonlinearly, by imposing the constraint (30). However, as we have shown, once the σ model is introduced by means of non-commutative geometry, the constraint is introduced naturally. This may explore why nonlinear σ model is realistic model.

At last, we emphasize that as the Higgs fields, the σ model has its geometrical origin. Similarly, other effective Lagrangian models can be studied in terms of non-commutative geometry also. We will discuss these topics in the future publications.

Acknowledgements

The author would like to thank H.B. Fei and Y. K. Lau for helpful discussions.

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